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The Ward property for a $\mathcal{P}$-adic basis and the $\mathcal{P}$-adic integral

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Abstract

An Henstock–Kurzweil type integral with respect to a $\mathcal{P}$-adic basis is considered. It is shown that a $\mathcal{P}$-adic basis possesses the Ward property if and only if the sequence by which it is defined is bounded. As a consequence, some full descriptive characterizations of the $\mathcal{P}$-adic integral in the bounded case are obtained. Moreover, an example of an exact $\mathcal{P}$-adic primitive which is not a *VBG* function and does not satisfy the Lusin condition (N) is constructed.

Keywords: $\mathcal{P}$-adic basis; $\mathcal{P}$-derivative; Ward property; Variational measure; Henstock–Kurzweil integral; *VBG* function

1. Introduction

In this paper we investigate some properties of the Henstock–Kurzweil type integral defined by a given $\mathcal{P}$-adic derivation basis. In the particular case of the sequence $\mathcal{P} = \{2, 2, \ldots, 2, \ldots\}$ we get the well known dyadic integral, which was studied in many papers and which has important applications in harmonic analysis, theory of martingales and some other areas of analysis (see [4,10,12,15,19–22]). More general $\mathcal{P}$-adic bases are also widely

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used in harmonic analysis, especially in connection with the group of characters of the respective $\mathcal{P}$-adic Cantor group (see [1,9,23]).

In Section 3 we present some basic properties of the $\mathcal{P}$-adic integral. In particular we formulate its partial descriptive characterization, as for the case of the integrals defined by some other bases (see [5,16]).

In Sections 4 and 5, in order to get its full descriptive characterization, we study the Ward property of an additive function defined on the intervals of a $\mathcal{P}$-adic basis. We show that a $\mathcal{P}$-adic basis possesses the Ward property if and only if the sequence $\mathcal{P}$ is bounded. In this case, as a consequence, we formulate a full descriptive characterization of the $\mathcal{P}$-adic integral.

In the last Section 6, we study relationships between the $\mathcal{P}$-adic Newton primitives and the classical *ACG* and *VBG* functions. In the particular case of the triadic basis we construct an example of an exact $\mathcal{P}$-adic primitive which is not a *VBG* function and which does not satisfy the Lusin condition (N).

2. Preliminaries

Let $\mathcal{P} = \{p_j\}_{j=0}^{\infty}$ be a fixed sequence of integers, with $p_j > 1$ for $j \in \mathbb{N} \cup \{0\}$. We set $m_0 = 1$ and $m_k = p_0 p_1 \dots p_{k-1}$, for $k \geqslant 1$. For fixed $k \in \mathbb{N} \cup \{0\}$, the closed intervals

$$\left[\frac{r}{m_k}, \frac{r+1}{m_k}\right] = I_r^{(k)}, \quad r = 0, 1, \dots, m_k - 1,$$

are called the *$\mathcal{P}$-adic intervals* (or simply the *$\mathcal{P}$-intervals*) of $[0,1]$ *of rank k*.

We denote by $\mathcal{I}_{\mathcal{P}}$ the set of all $\mathcal{P}$-intervals.

The points r/m_k, where $r = 0, 1, \dots, m_k$ and $k \in \mathbb{N} \cup \{0\}$, are called the *$\mathcal{P}$-adic rational points* of $[0,1]$. If $x \in [0,1]$ is not a $\mathcal{P}$-adic rational point, then it is called a *$\mathcal{P}$-adic irrational* point of $[0,1]$. For each $\mathcal{P}$-adic irrational point x, there exists only one $\mathcal{P}$-interval $I_x^{(k)} = [a_k(x), b_k(x)]$ of rank k containing x. Moreover, $\{x\} = \bigcap_{k=0}^{\infty} [a_k(x), b_k(x)]$, then we say that the sequence $\{[a_k(x), b_k(x)]\}$ of nested $\mathcal{P}$-intervals is the *basic sequence of $\mathcal{P}$-intervals convergent to x*. If x is a $\mathcal{P}$-adic rational point different from 0 and 1, then there exist two decreasing sequences of $\mathcal{P}$-intervals for which, starting with some k, x is a common end-point. Then for such a point we have two basic sequences convergent to x: the left one and the right one.

If $E \subset \mathbb{R}$ then $|E|$, $|E|_e$, $\bar{E}$ and $\operatorname{int} E$ denote the Lebesgue measure, the outer Lebesgue measure, the closure, and the interior of E, respectively. Almost everywhere (briefly a.e.) is always used in the sense of the Lebesgue measure. A positive function δ on $E \subset [0,1]$ is called a *gauge* on E.

For a given gauge δ on $E \subset [0,1]$ we set

$$\mathcal{P}_\delta[E] = \bigl\{(I, x) \colon I \in \mathcal{I}_{\mathcal{P}},\ x \in I \subset \bigl(x - \delta(x), x + \delta(x)\bigr),\ x \in E\bigr\}.$$

In the case $E = [0,1]$ we write simply $\mathcal{P}_\delta$. The family $\{\mathcal{P}_\delta\}_\delta$, where δ runs over the set of all gauges on $[0,1]$, is called the *$\mathcal{P}$-adic basis* on $[0,1]$. It is a differentiation basis in the sense of [4], [24], and [25].

Note that, if x is a $\mathcal{P}$-adic irrational point, then $\mathcal{P}_\delta[\{x\}] = \{(I_x^{(k)}, x)\colon I_x^{(k)} \subset (x-\delta(x), x+\delta(x))\}$; while for a $\mathcal{P}$-adic rational point x, the family $\mathcal{P}_\delta[\{x\}]$ is defined in a similar way by the left and the right basic sequences of $\mathcal{P}$-intervals convergent to x.

A finite subset π of $\mathcal{P}_\delta[E]$ is called a *$\mathcal{P}_\delta$-partition on E* if for distinct elements (I', x') and (I'', x'') in π, the intervals I' and I'' are nonoverlapping. If $\bigcup_{(I,x)\in\pi} I = E$ then we say that π is a partition *of E*.

It is easy to check that for any $\mathcal{P}$-interval I and for any gauge δ on I there exists a $\mathcal{P}_\delta$-partition of I.

A real interval function defined on $\mathcal{I}_\mathcal{P}$ is called a *$\mathcal{P}$-interval function*. We say that a $\mathcal{P}$-interval function Ψ is *additive* if $\Psi(I) = \sum_{i=1}^n \Psi(I_i)$, when the I_i, $i = 1, \ldots, n$, are nonoverlapping $\mathcal{P}$-intervals and $I = \bigcup_{i=1}^n I_i$. Moreover, we shall consider the extension of this function Ψ to the algebra generated by the family $\mathcal{I}_\mathcal{P}$.

A $\mathcal{P}$-interval function Ψ is said to be *$\mathcal{P}$-continuous* (or continuous with respect to the $\mathcal{P}$-adic basis) *at a point* x if for any $\varepsilon > 0$ there exists $\eta > 0$ such that $|\Psi(I)| < \varepsilon$ whenever $(I, x) \in \mathcal{P}_\eta[\{x\}]$.

The *upper* and the *lower derivatives* of a $\mathcal{P}$-interval function Ψ at a point x with respect to the $\mathcal{P}$-adic basis are defined respectively as

$$\overline{D}_\mathcal{P}\Psi(x) = \inf_\delta \sup\left\{\frac{\Psi(I)}{|I|}\colon (I,x) \in \mathcal{P}_\delta\big[\{x\}\big]\right\}$$

and

$$\underline{D}_\mathcal{P}\Psi(x) = \sup_\delta \inf\left\{\frac{\Psi(I)}{|I|}\colon (I,x) \in \mathcal{P}_\delta\big[\{x\}\big]\right\}.$$

If $\underline{D}_\mathcal{P}\Psi(x) = \overline{D}_\mathcal{P}\Psi(x) \neq \pm\infty$, we say that Ψ is *$\mathcal{P}$-differentiable* at x and the $\mathcal{P}$-derivative is denoted by $D_\mathcal{P}\Psi(x)$.

If Φ is a point function on $[0, 1]$ and $I = [c, d]$ is a $\mathcal{P}$-interval, we set

$$\Delta\Phi(I) = \Phi(d) - \Phi(c).$$

The function $\Delta\Phi$ is called the $\mathcal{P}$-interval function associated with Φ.

Remark 1. In what follows we shall often ascribe to a point function Φ the same properties that we have defined for an interval function, meaning that these properties are referred to the $\mathcal{P}$-interval function $\Delta\Phi$. For example, if we say that a point function Φ is $\mathcal{P}$-differentiable, then we mean that $\Delta\Phi$ is $\mathcal{P}$-differentiable. We shall also agree to use the notation $D_\mathcal{P}\Phi(x)$ for the $\mathcal{P}$-derivative of $\Delta\Phi$ at x, and we call it the $\mathcal{P}$-derivative of Φ at x.

3. Basic properties of the $\mathcal{P}$-adic integral

Let $\mathcal{P} = \{p_j\}_{j=0}^\infty$ be a fixed sequence of integers, with $p_j > 1$ for $j \in \mathbb{N} \cup \{0\}$.

Definition 3.1. A point function f on a $\mathcal{P}$-interval $[a,b] \subset [0,1]$ is said to be *integrable with respect to the $\mathcal{P}$-adic basis* (or simply *$\mathcal{P}$-integrable*) on $[a,b]$, with $\mathcal{P}$-adic integral (or simply $\mathcal{P}$-integral) A, if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - A \right| < \varepsilon,$$

for any partition $\pi \subset \mathcal{P}_\delta$ of $[a,b]$. We write $A = (\mathcal{P})\int_a^b f$.

For this integral many of the usual properties hold (for the Henstock–Kurzweil integral defined by a general basis see [14] and [24]). In particular:

Proposition 3.2. *If a function f is $\mathcal{P}$-integrable on a $\mathcal{P}$-interval $[a,b]$, then it is $\mathcal{P}$-integrable on each $\mathcal{P}$-interval $I \subset [a,b]$. Moreover, the $\mathcal{P}$-interval function $F(I) = (\mathcal{P})\int_I f$ is additive.*

We call $F(I) = (\mathcal{P})\int_I f$ the indefinite $\mathcal{P}$-adic integral (or simply the indefinite $\mathcal{P}$-integral) of f on $[a,b]$.

Proposition 3.3. *If a $\mathcal{P}$-interval function F is $\mathcal{P}$-differentiable everywhere on a $\mathcal{P}$-interval $[a,b]$, then its $\mathcal{P}$-derivative f is $\mathcal{P}$-integrable on $[a,b]$ and F is the indefinite $\mathcal{P}$-integral of f on $[a,b]$.*

Proposition 3.4. *If a function f is $\mathcal{P}$-integrable on $[a,b]$, then its indefinite $\mathcal{P}$-integral F is $\mathcal{P}$-continuous at each point of $[a,b]$. Moreover, F is $\mathcal{P}$-differentiable almost everywhere on $[a,b]$, with $D_{\mathcal{P}}F(x) = f(x)$ almost everywhere.*

Now, in order to give a descriptive characterization of the indefinite $\mathcal{P}$-integral, we use the notion of variational measure generated by a $\mathcal{P}$-interval function.

Given a $\mathcal{P}$-interval function Ψ, a set $E \subset [0,1]$ and a gauge δ on E, we define the *δ-variation of Ψ on E* as

$$\mathrm{Var}(\mathcal{P}_\delta, \Psi, E) = \sup \sum_{(I,x)\in\pi} \left|\Psi(I)\right|,$$

where the supremum is taken over all partitions $\pi \subset \mathcal{P}_\delta[E]$. Then we define the *$\mathcal{P}$-variational measure generated by Ψ* as

$$V_\Psi(E) = \inf \mathrm{Var}(\mathcal{P}_\delta, \Psi, E), \tag{1}$$

where the infimum is taken over all gauges δ on E. Note that V_Ψ is a metric outer measure on $[0,1]$ (see [25]) and so its restriction to the Borel sets is a measure.

We recall that a measure μ is said to be *absolutely continuous* with respect to the Lebesgue measure if $|N| = 0$ implies $\mu(N) = 0$.

The following theorem can be proved by standard methods (see [5] for the ordinary interval basis).

Theorem 3.5. *An additive $\mathcal{P}$-interval function F is the indefinite $\mathcal{P}$-integral of a function f on $[0,1]$ if and only if F generates an absolutely continuous $\mathcal{P}$-variational measure and F is $\mathcal{P}$-differentiable almost everywhere on $[0,1]$, with $D_{\mathcal{P}}F(x) = f(x)$ almost everywhere.*

To formulate the next characterization of the $\mathcal{P}$-integral we introduce the notion of $\mathcal{P}ACG_\delta$ function.

Definition 3.6. A $\mathcal{P}$-interval function F is said to be $\mathcal{P}AC_\delta$ on $E \subset [0,1]$ if for any $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ on E such that $\sum_{(I,x)\in\pi} |F(I)| < \varepsilon$, for any partition $\pi \subset \mathcal{P}_\delta[E]$ with $\sum_{(I,x)\in\pi} |I| < \eta$. A $\mathcal{P}$-interval function F is said to be $\mathcal{P}ACG_\delta$ on E if E can be written as a countable union of sets on each of which F is $\mathcal{P}AC_\delta$.

The following statement can be proved as Theorem 9.17 in [11].

Theorem 3.7. *An additive $\mathcal{P}$-interval function F is the indefinite $\mathcal{P}$-integral of a function f on $[0,1]$ if and only if F is $\mathcal{P}ACG_\delta$ on $[0,1]$ and F is $\mathcal{P}$-differentiable almost everywhere on $[0,1]$, with $D_{\mathcal{P}}F(x) = f(x)$ almost everywhere.*

The descriptive characterizations of the $\mathcal{P}$-integral given by Theorems 3.5 and 3.7 are called *partial descriptive characterizations* since they require the differentiability a.e. of the interval function F. A descriptive characterization of the $\mathcal{P}$-integral is called a *full descriptive characterization* if no differentiability assumptions is supposed a priori. In what follows we show that the $\mathcal{P}$-integral admits full descriptive characterizations only in the case the sequence $\mathcal{P}$ is bounded.

4. The Ward property and the full descriptive characterization in the bounded case

We say that a given differentiation basis $\mathcal{B}$ satisfies the Ward property whenever *each additive interval function is $\mathcal{B}$-differentiable almost everywhere on the set of all points at which at least one of its extreme $\mathcal{B}$-derivatives is finite*.

The Ward property of a basis plays an important role in obtaining a descriptive characterization of the integral defined by the basis (see [2–4,6,7]).

Here we prove the following theorem.

Theorem 4.1. *The $\mathcal{P}$-adic basis has the Ward property if and only if the sequence $\mathcal{P}$ is bounded.*

The necessity part will be proved in Section 5. The sufficient part can be deduced from some general results of [17] related to derivatives of cell functions in an abstract space, or can be proved by adjusting to our case the original proof of Ward (see [18]). In the last case it is enough to use the following lemma, instead of Lemma 11.8 in Chapter IV of [18], where it was formulated for the ordinary interval basis.

Lemma 4.2. *Let $\mathcal{P}$ be a bounded sequence and set $p = \sup_j p_j$. Suppose that G is an additive $\mathcal{P}$-interval function, I is a fixed $\mathcal{P}$-interval, E is a subset of I, and $0 < \varepsilon < 1/p$ and α are real constants such that*

(i) $|E|_e > (1-\varepsilon)|I|$;
(ii) $G(J) > 0$, *for any $\mathcal{P}$-interval $J \subset I$ with $J \cap E \neq \emptyset$*;
(iii) $\overline{D}_{\mathcal{P}} G(x) > \alpha$, *at any point $x \in E$.*

Then

$$G(I) > \alpha(1 - p\varepsilon)|I|.$$

The proof is very similar to that of [18, Chapter IV, Lemma 11.8], then we omit it.

In [4] we proved the following theorem.

Theorem A. *Let $\mathcal{B}$ be a differentiation basis having the Busemann–Feller property and the Ward property. Then any additive $\mathcal{B}$-interval function F is an indefinite $\mathcal{B}$-Henstock–Kurzweil integral if and only if F generates an absolutely continuous $\mathcal{B}$-variational measure.*

Now our $\mathcal{P}$-adic basis satisfies the Busemann–Feller property, since for each $k \in \mathbb{N} \cup \{0\}$, each $r = 0, 1, \ldots, m_k - 1$, and each $x \in I_r^{(k)}$ there exists $\delta > 0$ such that $(I_r^{(k)}, x) \in \mathcal{P}_\delta$.

Then, by Theorem 4.1 and Theorem A, and using also Theorem 3.5, we get the following full descriptive characterization of the $\mathcal{P}$-integral.

Theorem 4.3. *Let $\mathcal{P}$ be a bounded sequence. An additive $\mathcal{P}$-interval function F is the indefinite $\mathcal{P}$-integral of a function f on* $[0, 1]$ *if and only if F generates an absolutely continuous $\mathcal{P}$-variational measure. In such a case $D_{\mathcal{P}} F(x) = f(x)$ almost everywhere on* $[0, 1]$.

It is easy to check that each $\mathcal{P}ACG_\delta$ function generates an absolutely continuous $\mathcal{P}$-variational measure. So, by Theorems 3.7 and 4.3, we obtain the following version of the full descriptive characterization of the $\mathcal{P}$-integral.

Theorem 4.4. *Let $\mathcal{P}$ be a bounded sequence. An additive $\mathcal{P}$-interval function F is the indefinite $\mathcal{P}$-integral of a function f on* $[0, 1]$ *if and only if F is $\mathcal{P}ACG_\delta$ on* $[0, 1]$. *In such a case $D_{\mathcal{P}} F(x) = f(x)$ almost everywhere on* $[0, 1]$.

5. On the Ward property in the unbounded case

An example of a basis which does not possess the Ward property was constructed in [17]. We show here that a $\mathcal{P}$-adic basis defined by any unbounded sequence $\mathcal{P}$ fails to possess the Ward property. This proves the necessity part of Theorem 4.1.

Theorem 5.1. *For any unbounded sequence $\mathcal{P} = \{p_j\}_{j=0}^{\infty}$ there exist a closed set S of positive measure and*

(i) *a continuous point function F on $[0,1]$, piecewise linear on each interval contiguous to S, and such that*

$$\overline{D}_{\mathcal{P}} F(x) = +1 \quad \text{and} \quad \underline{D}_{\mathcal{P}} F(x) = -1 \tag{2}$$

for any $\mathcal{P}$-adic irrational point $x \in S$;

(ii) *a continuous point function G on $[0,1]$ such that*

$$\overline{D}_{\mathcal{P}} G(x) = \underline{D}_{\mathcal{P}} G(x) = +\infty \tag{3}$$

for any $x \in S$.

Proof. As $\mathcal{P}$ is unbounded, we can find a subsequence $\{p_{k_j}\}_{j=1}^{\infty}$, such that $k_1 = 1$ and

$$\sum_{j=1}^{\infty} \frac{1}{p_{k_j}} < +\infty. \tag{4}$$

Note that

$$I_i^{(k)} = \bigcup_{j=ip_k}^{(i+1)p_k-1} I_j^{(k+1)}.$$

Now we fix an arbitrary $k \geqslant 1$ and we define the following continuous and piecewise linear function:

$$F_k(x) = \begin{cases} x - \alpha_i & \text{if } x \in I_i^{(k)} \setminus I_{(i+1)p_k-1}^{(k+1)},\ i = 0, \ldots, m_k - 1, \\ (p_k - 1)(\beta_i - x) & \text{if } x \in I_{(i+1)p_k-1}^{(k+1)},\ i = 0, \ldots, m_k - 1, \end{cases}$$

where, for each i, α_i and β_i are the end-points of the interval $I_i^{(k)}$. Put

$$P_k = \bigcup_{i=0}^{m_k-1} \overline{\left(I_i^{(k)} \setminus I_{(i+1)p_k-1}^{(k+1)}\right)},$$

$$S_j = \bigcap_{l=1}^{j} P_{k_l}, \quad \text{and} \quad S = \bigcap_{j=1}^{\infty} S_j = \bigcap_{j=1}^{\infty} P_{k_j}.$$

It is obvious that

$$|P_k| = 1 - \frac{1}{p_k}$$

and that $\{P_k\}$ is a sequence of independent sets. So

$$|S| = \prod_{j=1}^{\infty} \left(1 - \frac{1}{p_{k_j}}\right),$$

and by (4) the set S is of positive measure. Moreover, S is closed (even perfect) and the contiguous intervals of S are unions of those $\mathcal{P}$-adic intervals which are not included into P_{k_j}. Now set

$$F(x) = F_1(x) + \sum_{j=2}^{\infty} 2(-1)^{j+1} F_{k_j}(x)\chi_{S_j}(x). \tag{5}$$

Since, for $k = 1, 2, \dots,$ we have

$$\max_x \bigl|F_k(x)\bigr| = \bigl|I_i^{(k)} \setminus I_{(i+1)p_k-1}^{(k+1)}\bigr| \leqslant \frac{1}{m_k} \leqslant \frac{1}{2^k},$$

then the function F is continuous, as a sum of an uniformly convergent series of continuous functions. In particular, since the sum in (5) is finite at each point of any interval contiguous to S, then F is piecewise linear on these intervals.

From the definition of F_k it follows that

$$\Delta F_k(I_i^{(s)}) = 0 \quad \text{for all } s \leqslant k \tag{6}$$

and

$$\frac{\Delta F_k(I_i^{(s)})}{|I_i^{(s)}|} = 1 \quad \text{for all } s \geqslant k+1 \text{ with } I_i^{(s)} \subset P_k. \tag{7}$$

Now, for each natural s, we can find an unique t such that

$$k_t + 1 \leqslant s \leqslant k_{t+1}; \tag{8}$$

then by (6) we have

$$\Delta F\bigl(I_i^{(s)}\bigr) = \Delta F_1\bigl(I_i^{(s)}\bigr) + \sum_{j=2}^{t} 2(-1)^{j+1} \Delta F_{k_j}\bigl(I_i^{(s)}\bigr). \tag{9}$$

By the construction of S, for any $\mathcal{P}$-adic irrational point $x \in S$ there is an unique basic sequence $\{I_x^{(s)}\}$ of $\mathcal{P}$-intervals convergent to x. In particular, (9) holds for the intervals $I_x^{(s)}$ such that s satisfies condition (8). Then, using (7) and (9) for such an s, we have

$$\frac{\Delta F(I_x^{(s)})}{|I_x^{(s)}|} = 1 + \sum_{j=2}^{t} 2(-1)^{j+1}.$$

So, if s satisfies (8) with odd t, we have

$$\frac{\Delta F(I_x^{(s)})}{|I_x^{(s)}|} = 1, \tag{10}$$

and, if s satisfies (8) with even t, we have

$$\frac{\Delta F(I_x^{(s)})}{|I_x^{(s)}|} = -1. \tag{11}$$

This implies (2) and completes the proof of part (i) of the theorem.

To prove part (ii), we define

$$G(x) = F_1(x) + \sum_{j=2}^{\infty} F_{k_j}(x)\chi_{S_j}(x).$$

Then, for each $\mathcal{P}$-adic irrational point $x \in S$ and for s satisfying (8), we have

$$\Delta G\big(I_i^{(s)}\big) = \Delta F_1\big(I_i^{(s)}\big) + \sum_{j=2}^{t} \Delta F_{k_j}\big(I_i^{(s)}\big),$$

and

$$\frac{\Delta G(I_x^{(s)})}{|I_x^{(s)}|} = 1 + \sum_{j=2}^{t} 1 = t.$$

This gives (3) and completes the proof of part (ii) of the theorem. □

If $\mathcal{P}$ is unbounded, by Theorem 5.1 we cannot get analogues of Theorems 4.3 and 4.4.

Theorem 5.2. *For any unbounded sequence $\mathcal{P}$ there exists on* $[0,1]$ *an additive continuous $\mathcal{P}ACG_\delta$ function which is not an indefinite $\mathcal{P}$-integral.*

Proof. The required function is given by the function ΔF associated with the point function F of Theorem 5.1. Indeed, from the construction of F (see (10) and (11)) it follows that for any $x \in S$ and for any $\mathcal{P}$-interval $I \ni x$ we have $|\Delta F(I)| = |I|$. So, for arbitrary $\varepsilon > 0$, we can take $\eta = \varepsilon$ in the definition of $\mathcal{P}AC_\delta$. Then for any gauge δ on S and for any partition $\pi \subset \mathcal{P}_\delta[S]$ such that $\sum_{(I,x)\in\pi} |I| < \eta$, we get $\sum_{(I,x)\in\pi} \Delta F(I) < \varepsilon$. Hence F is $\mathcal{P}AC_\delta$ on S (see Remark 1). Since F is piecewise linear on each interval contiguous to S, it is obviously $\mathcal{P}ACG_\delta$ on the complement of S, and therefore $\mathcal{P}ACG_\delta$ on the whole interval $[0,1]$. On the other hand, F is not $\mathcal{P}$-differentiable on the set S of positive measure. So, by Proposition 3.4, F is not an indefinite $\mathcal{P}$-integral. □

As we have already mentioned, each $\mathcal{P}ACG_\delta$ function generates an absolutely continuous $\mathcal{P}$-variational measure. So, with the above example, we have also established the following theorem.

Theorem 5.3. *For any unbounded sequence $\mathcal{P}$ there exists an additive continuous function F which generates an absolutely continuous $\mathcal{P}$-variational measure and which is not an indefinite $\mathcal{P}$-integral.*

Denoting by $\mathcal{P}HK$ the class of all additive $\mathcal{P}$-interval functions which are indefinite $\mathcal{P}$-integrals and by $\mathcal{P}ACVM$ the class of all additive functions generating an absolutely continuous $\mathcal{P}$-variational measure, we can summarize our results related to any unbounded sequence $\mathcal{P}$ in the following scheme:

$$\mathcal{P}HK \subsetneqq \mathcal{P}ACG_\delta \subset \mathcal{P}ACVM.$$

Whether the last inclusion is proper or not for a $\mathcal{P}$-adic basis such that the sequence $\mathcal{P}$ is unbounded, it is an open problem.

6. $\mathcal{P}$-adic primitives and *VBG*-functions

In this section we consider some properties of the class of all *exact $\mathcal{P}$-adic primitives* (i.e., functions having finite $\mathcal{P}$-derivative everywhere). Following [18], we call the integral defined by such primitives the *Newton $\mathcal{P}$-adic integral* (or simply the *Newton $\mathcal{P}$-integral*). So the exact $\mathcal{P}$-adic primitives (or simply the exact $\mathcal{P}$-primitives) are the class of all indefinite integrals in this sense. Here we study the relationships between the Newton $\mathcal{P}$-integral and the classical *ACG* and *VBG* notions (for the definitions see [18]).

In [20] it is shown that in the dyadic case (i.e., $\mathcal{P} = \{2, 2, \dots, 2, \dots\}$), the indefinite $\mathcal{P}$-integral can fail to be a *VBG* function and then to be an *ACG* function. On the other hand, in [19] it is proved that all the exact dyadic primitives are (ACG) functions (we remind that a function F is said to be (ACG) on E if E can be written as a countable union of sets on each of which F is *AC*. So an (ACG) function is not supposed to be continuous).

Here we show that the last mentioned property of the Newton indefinite dyadic integral is rather an exception. Indeed, already in the case of the triadic basis (i.e., $\mathcal{P} = \{3, 3, \dots, 3, \dots\}$), the exact $\mathcal{P}$-primitive can fail to be a *VBG* function and then to be an (ACG) function.

Theorem 6.1. *There exists a continuous function F on $[0, 1]$ which is differentiable everywhere with respect to the triadic basis and which is not a VBG function.*

Proof. To construct an example proving the claim we shall use some ideas and notations of [20], adjusting them to the triadic case.

By $I_r^{(k)}$ we denote below exclusively the triadic intervals; i.e.,

$$I_r^{(k)} = \left[\frac{r}{3^k}, \frac{r+1}{3^k}\right].$$

We set

$$Q_1 = I_1^{(3)} \cup I_7^{(3)} \cup I_{19}^{(3)} \cup I_{25}^{(3)} \quad \text{and} \quad T_1 = I_1^{(1)} \cup I_1^{(2)} \cup I_7^{(2)} \cup \big(\mathbb{R} \setminus (0, 1)\big),$$

and we define on $\mathbb{R}$ an auxiliary function Φ, by putting $\Phi(x) = 1$ if $x \in Q_1$, $\Phi(x) = 0$ if $x \in T_1$, and $\Phi(x)$ linear on the closure of each interval of the complement of $Q_1 \cup T_1$.

Note that

$$0 \leqslant \Phi(x) \leqslant 1, \quad \text{for all } x \in \mathbb{R}, \tag{12}$$

and that

$$\Delta\Phi\big(I_x^{(k)}\big) = 0, \quad k = 0, 1, \dots, \tag{13}$$

for each $x \in \operatorname{int} Q_1$.

For a given interval $U = (\alpha, \beta)$ we denote by $L(x, U)$ the following linear mapping on $\mathbb{R}$:

$$L(x, U) = |U|^{-1}(x - \alpha). \tag{14}$$

We define by induction a double sequence of triadic intervals $\{U_j^{(n)}\}$, $j = 0, 1, \dots, 4^n - 1$, $n \in \mathbb{N}$. For $n = 1$ we put

$$U_0^{(1)} = I_1^{(3)}, \qquad U_1^{(1)} = I_7^{(3)}, \qquad U_2^{(1)} = I_{19}^{(3)}, \qquad U_3^{(1)} = I_{25}^{(3)}.$$

Now suppose $U_j^{(n)}$ is defined for some $n \geqslant 1$ and each $j = 0, 1, \dots, 4^n - 1$. Then we define $U_{4j+i}^{(n+1)}$, $i = 0, 1, 2, 3$, to be the inverse image under $L(x, U_j^{(n)})$ of the intervals $I_1^{(3)}$, $I_7^{(3)}$, $I_{19}^{(3)}$, $I_{25}^{(3)}$, respectively. Thus we have defined $U_j^{(n+1)}$ for all $j = 0, 1, \dots, 4^{n+1} - 1$.

Obviously, for a fixed n the intervals $U_j^{(n)}$, $0 \leqslant j \leqslant 4^n - 1$, are disjoint triadic intervals of rank $3n$.

Put

$$Q_n = \bigcup_{j=0}^{4^n-1} U_j^{(n)}, \quad n = 1, 2, \dots, \quad \text{and} \quad Q = \bigcap_{n=1}^{\infty} Q_n. \tag{15}$$

Observe that each point of Q is a triadic irrational. We have $|Q_1| = 4/27$. Hence

$$|Q_n| = \left(\frac{4}{27}\right)^n,$$

and so $|Q| = 0$.

Now we consider the following double sequence of functions

$$\Phi_j^{(n)}(x) = \Phi\bigl(L\bigl(x, U_j^{(n)}\bigr)\bigr), \quad n = 1, 2, \dots, \; j = 0, 1, \dots, 4^n - 1, \; x \in \mathbb{R}.$$

It can be easily deduced from (12), (13) and from the definitions of $U_j^{(n)}$ and $\Phi_j^{(n)}$, that the functions $\Phi_j^{(n)}$ have the following properties:

$$0 \leqslant \Phi_j^{(n)}(x) \leqslant 1 \quad \text{for } x \in \mathbb{R}, \tag{16}$$

$$\Phi_j^{(n)}(x) = 0 \quad \text{if } x \notin U_j^{(n)}, \tag{17}$$

$$\Phi_j^{(n)}(x) = 1 \quad \text{if } x \in Q_{n+1}, \tag{18}$$

$$\Delta\Phi_j^{(n)}(I_x^{(m)}) = 0 \quad \text{for } m \geqslant 0 \text{ and } x \in \operatorname{int} Q_{n+1}. \tag{19}$$

In correspondence to the above sequence of functions, we also define by induction a double sign sequence $\{s_j^{(n)}\}$ with terms ± 1.

We start with $s_i^{(1)} = (-1)^{i+1}$, $i = 0, 1, 2, 3$. Then the inductive step from n to $n+1$ is defined by

$$\begin{aligned} s_{4j+i}^{(n+1)} &= (-1)^i s_j^{(n)}, \quad \text{if } j = 4p \text{ or if } j = 4p+2 \text{ and } s_j^{(n)} = s_{j-1}^{(n)}, \\ &= (-1)^{i+1} s_j^{(n)}, \quad \text{if } j = 4p+3 \text{ or if } j = 4p+1 \text{ and } s_j^{(n)} = s_{j+1}^{(n)}, \\ &= (-1)^{i+1+[i/2]} s_j^{(n)}, \quad \text{if } j = 4p+1 \text{ and } s_j^{(n)} = -s_{j+1}^{(n)}, \\ &= (-1)^{i+[i/2]} s_j^{(n)}, \quad \text{if } j = 4p+2 \text{ and } s_j^{(n)} = -s_{j-1}^{(n)}, \end{aligned} \tag{20}$$

where $j = 0, 1, \dots, 4^n - 1$, $i = 0, 1, 2, 3$, and $[i/2]$ is the integer part of $i/2$.

Observe that, by (20), $s_{4j}^{(n+1)} = s_j^{(n)}$ if $j = 4p$ and hence

$$s_{4^{k+1}j}^{(n+k)} = s_{4j}^{(n)} \quad \text{for } j = 0, 1, \dots, 4^{n-1} - 1 \text{ and any } k \geqslant 0. \tag{21}$$

Similarly, by (20), $s_{4j+3}^{(n+1)} = s_j^{(n)}$ if $j = 4p+3 = 4(p+1)-1$ and hence

$$s_{4^{k+1}j-1}^{(n+k)} = s_{4j-1}^{(n)} \quad \text{for } j = 0, 1, \ldots, 4^{n-1}-1 \text{ and any } k \geqslant 0. \tag{22}$$

Now, restricting all the functions $\Phi_j^{(n)}$ to the interval [0, 1], we put

$$F_n(x) = \frac{1}{2}\Phi(x) + \sum_{k=1}^{n} \sum_{j=0}^{4^k-1} \frac{1}{2^{k+1}} s_j^{(k)} \Phi_j^{(k)}(x) \tag{23}$$

and

$$F(x) = \lim_{n\to\infty} F_n(x) = \frac{1}{2}\Phi(x) + \sum_{k=1}^{\infty} \sum_{j=0}^{4^k-1} \frac{1}{2^{k+1}} s_j^{(k)} \Phi_j^{(k)}(x). \tag{24}$$

As for a fixed k the intervals $U_j^{(k)}$ supporting the functions $\Phi_j^{(k)}$ are disjoint, it follows from (16) that the series in (24) converges uniformly. Therefore F is continuous on [0, 1]. In particular, since for each $x \notin Q$ the sum in (24) is finite, then F is piecewise linear on each interval contiguous to Q. Hence $F'(x) = D_{\mathcal{P}} F(x)$ exists everywhere outside Q, except at some triadic rational points.

Now fix $x \in Q$ (remember that such a x is a triadic irrational point). Then for any triadic interval of the basic sequence $\{I_x^{(m)}\}$, by (24) and (19), we get $\Delta F(I_x^{(m)}) = 0$. This implies

$$D_{\mathcal{P}} F(x) = 0 \quad \text{if } x \in Q.$$

Without affecting the values of F on Q and the values of $D_{\mathcal{P}} F$ on Q, it is not difficult to smooth the function F at the angular points outside Q to make F to be differentiable everywhere outside Q. Therefore we can assume that F is $\mathcal{P}$-differentiable everywhere on [0, 1].

Now we prove that the function F is not *VBG*. We compute first the values of F at the end-points of each interval contiguous to Q. Notice that

$$F_n(x) = F_{n-1}(x) + \sum_{j=0}^{4^n-1} \frac{1}{2^{n+1}} s_j^{(n)} \Phi_j^{(n)}(x).$$

Then F_n is constant on each $U_{4j+i}^{(n+1)}$. We denote this constant by $F_n(U_{4j+i}^{(n+1)})$. We remark that, for a fixed j and for $i = 0, 1, 2, 3$, by (18) we have

$$F_n\big(U_{4j+i}^{(n+1)}\big) = F_{n-1}\big(U_j^{(n)}\big) + \frac{1}{2^{n+1}} s_j^{(n)}. \tag{25}$$

Now consider two neighboring intervals $U_{j-1}^{(n)}$ and $U_j^{(n)}$ lying on the same fixed interval $U_p^{(n-1)}$, $p = 0, 1, \ldots, 4^{n-1}-1$. Then we have $j = 4p+m$, $m = 1, 2, 3$, and

$$F_{n-1}\big(U_{j-1}^{(n)}\big) = F_{n-1}\big(U_j^{(n)}\big). \tag{26}$$

Note that for any $k \geqslant 1$ the interval $U_{4^k j-1}^{(n+k)}$ is the furthest right of $U_i^{(n+k)}$ on $U_{j-1}^{(n)}$ and the interval $U_{4^k j}^{(n+k)}$ is the furthest left of $U_i^{(n+k)}$ on $U_j^{(n)}$.

If

$$s_{j-1}^{(n)} = -s_j^{(n)}, \tag{27}$$

then, considering all the possible cases in (20), we get

$$s_{4j-1}^{(n+1)} = -s_{j-1}^{(n)} \quad \text{and} \quad s_{4j}^{(n+1)} = -s_j^{(n)}. \tag{28}$$

If

$$s_{j-1}^{(n)} = s_j^{(n)} \tag{29}$$

(that, according to (20), could occur only if $j = 4p + 2$ and $j - 1 = 4p + 1$), then we get

$$s_{4j-1}^{(n+1)} = s_j^{(n)} \quad \text{and} \quad s_{4j}^{(n+1)} = s_j^{(n)}. \tag{30}$$

Denoting

$$\alpha = \bigcap_{k=0}^{\infty} U_{4^k j-1}^{(n+k)}, \qquad \beta = \bigcap_{k=0}^{\infty} U_{4^k j}^{(n+k)}, \tag{31}$$

by (15) we have $\alpha, \beta \in Q$. Since $(\alpha, \beta) \cap Q$ is empty, then (α, β) is an interval contiguous to Q. Moreover, $(\alpha, \beta) \subset U_p^{(n-1)}$.

In the case (27), using (18), (21), (22), (24), (25) and (28), we compute

$$F(\alpha) = F_{n-1}\big(U_{j-1}^{(n)}\big) + s_{j-1}^{(n)}\left(\frac{1}{2^{n+1}} - \sum_{k=n+2}^{\infty} \frac{1}{2^k}\right) = F_{n-1}\big(U_{j-1}^{(n)}\big),$$

$$F(\beta) = F_{n-1}\big(U_j^{(n)}\big) + s_j^{(n)}\left(\frac{1}{2^{n+1}} - \sum_{k=n+2}^{\infty} \frac{1}{2^k}\right) = F_{n-1}\big(U_j^{(n)}\big).$$

By (26), this implies

$$F(\alpha) = F(\beta) = F_{n-1}\big(U_{j-1}^{(n)}\big). \tag{32}$$

Note that the last equality always holds if $j = 4p + 1$ or $j = 4p + 3$.

Repeating the same argument in the case (29) and using (30) instead of (28), we get

$$F(\alpha) = F(\beta) = F_{n-1}\big(U_j^{(n)}\big) + s_j^{(n)}\left(\frac{1}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{1}{2^k}\right) = F_{n-1}\big(U_j^{(n)}\big) + s_j^{(n)}\frac{1}{2^n}.$$

It is easy to check that for any interval (α, β), proper contiguous to Q, its end-points α and β can be represented in the form (31) where $j = 4p + 1$, or $j = 4p + 2$, or $j = 4p + 3$. Therefore we have proved that for any interval (α, β) proper contiguous to Q, the equality

$$F(\alpha) = F(\beta) \tag{33}$$

holds.

It is also easy to prove that for two improper contiguous intervals $(0, \beta)$ and $(\alpha, 1)$ we have

$$F(0) = F(\beta) = 0, \qquad F(\alpha) = F(1) = 1. \tag{34}$$

It is important to observe that by (32) (with $j = 4p + 1$ and n replaced by $n + 1$) and by (31), any interval $U_p^{(n)}$ contains a point $x \in Q$ such that $F(x) = F_n(U_{4p}^{(n+1)})$.

Now consider intervals $U_j^{(n)} \subset U_p^{(n-1)}$ with $j = 4p + m$, $m = 0, 1, 2, 3$. Since $s_j^{(n)}$ changes sign at least twice when j runs from $4p$ to $4p + 3$ (see (20)), then by (18), (25), (26), and by the above observation we get

$$V\big(F, Q \cap U_p^{(n-1)}\big) \geqslant \frac{1}{2^n},$$

where V stands for the classical variation of F on a set. Since this holds for any n, and each $U_j^{(n)}$ contains 4^k intervals $U_i^{(n+k)}$, $4^k j \leqslant i \leqslant 4^k(j + 1) - 1$, then

$$V\big(F, Q \cap U_p^{(n)}\big) \geqslant \sum_{i=4^k j}^{4^k(j+1)-1} V\big(F, Q \cap U_i^{(n+k)}\big) \geqslant 4^k \frac{1}{2^{n+k+1}} = 2^{k-n-1},$$

for arbitrary $k \geqslant 1$.

Since F is not of bounded variation on any portion of the set Q, then F is not *VBG* on Q and hence also on $[0, 1]$. This completes the proof. □

Now consider the function $F(x, Q) = F(x)$ on $Q \cup \{0, 1\}$ and linear on the closure of each interval contiguous to Q. By (33) and (34), $F(x, Q)$ is in fact constant on each interval contiguous to Q, and $F(0, Q) = 0$, $F(1, Q) = 1$. Therefore $F(x, Q)$ maps $[0, 1]$ on $[0, 1]$, and $[0, 1] \setminus Q$ on a countable set. So the image of the null set Q under the function $F(x, Q)$ is a set of positive measure, and by the definition of $F(x, Q)$ the same is true for the function F. This implies that F does not satisfy the Lusin condition (N).

On the other hand, since the function F is an exact $\mathcal{P}$-primitive then, by Proposition 3.3, F is the indefinite $\mathcal{P}$-integral of its $\mathcal{P}$-derivative. So, by Theorem 3.5, F generates an absolutely continuous $\mathcal{P}$-variational measure. Hence we have got the following result:

Theorem 6.2. *There exists a continuous function which generates an absolutely continuous $\mathcal{P}$-variational measure, but it is not VBG and does not satisfy the Lusin condition (N).*

Remark 2. In some literature (see [13]) a function generating an absolutely continuous variational measure is said to satisfy the *strong Lusin condition*. So the *strong Lusin condition with respect to a $\mathcal{P}$-adic basis does not imply the Lusin condition* (N), in contrast with the case of the usual interval basis and of some other bases (see [8]).

Remark 3. Another consequence of the above example is the fact that the Newton $\mathcal{P}$-integral, in the case of the triadic basis, and the Denjoy–Khintchine integral are noncompatible. Indeed, let us consider the function $H(x) = F(x) - F(x, Q)$. It is a continuous function equal to zero on Q and piecewise linear on each interval contiguous to Q. Then H is *ACG* and differentiable almost everywhere (in the ordinary sense) on $[0, 1]$. So H is the indefinite Denjoy–Khintchine integral of its derivative. Now, since $H'(x) = D_{\mathcal{P}}F(x)$ outside Q (i.e., almost everywhere on $[0, 1]$), then the function H is also the indefinite Denjoy–Khintchine integral of $D_{\mathcal{P}}F$. On the other hand, F is the Newton triadic integral of $D_{\mathcal{P}}F$. As the difference between those indefinite integrals is equal to $F(x, Q)$, which is not a constant, we get that there exists a function f on $[0, 1]$, which is both Newton triadic integrable and Denjoy–Khintchine integrable on $[0, 1]$, but the values of the two integrals are different. (As we have already mentioned, in the dyadic case the Newton $\mathcal{P}$-integral is (ACG) and hence it is compatible with the Denjoy–Khintchine integral.)

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